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Unprovability threshold for the planar graph minor theorem

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ABSTRACT

This note is part of the implementation of a programme in foundations of mathematics to find exact threshold versions of all mathematical unprovability results known so far, a programme initiated by Weiermann. Here we find the exact versions of unprovability of the finite graph minor theorem with growth rate condition restricted to planar graphs, connected planar graphs and graphs embeddable into a given surface, assuming an unproved conjecture (*): 'there is a number $a > 0$ such that for all $k \geq 3$, and all $n \geq 1$, the proportion of connected graphs among unlabelled planar graphs of size n omitting the k-element circle as minor is greater than a'. Let γ be the unlabelled planar growth constant ($27.2269 \leq \gamma < 30.061$). Let $P(c)$ be the following first-order arithmetical statement with real parameter c: "for every K there is N such that whenever $G_1, G_2, \ldots, G_N$ are unlabelled planar graphs with $|G_i| < K + c \cdot \log_2 i$ then for some $i < j \leq N$, G_i is isomorphic to a minor of G_j". Then

1. for every $c \leq \frac{1}{\log_2 \gamma}$, $P(c)$ is provable in $I\Delta_0 + \exp$;
2. for every $c > \frac{1}{\log_2 \gamma}$, $P(c)$ is unprovable in ATR_0.

We also give proofs of some upper and lower bounds for unprovability thresholds in the general case of the finite graph minor theorem.

A minor of a graph is a graph obtained from it by some sequence of edge-deletions, edge-contractions and deletions of isolated vertices. The graph minor theorem "for every infinite sequence $G_1, G_2, \ldots, G_i, \ldots$ of finite simple unlabelled graphs, there are $i < j$ such that G_i is isomorphic to a minor of G_j" was proved by N. Robertson and P. Seymour in a long series of papers in the 1980s. For a discussion of the graph minor theorem, see for example the textbook [5], chapter 12.

Later it was discovered by Friedman et al. [6] that this statement possesses an enormous amount of arithmetical strength. (The exact strength of the graph minor theorem is still an open problem in 2009. Π^1_1-CA_0 is a lower bound for the strength of the graph minor theorem [6], even in the case of the graph minor theorem restricted to graphs of bounded tree-width. Friedman conjectured in [6] that the full graph minor theorem is unprovable in Π^1_1-CA. The upper bound in [6] is Π^1_1-CA+BI.) The presence of strength in the graph minor theorem is of course in stark contrast with many other important mathematical theorems which happen to be provable in a weak system.

The first-order miniaturisation of the infinite graph minor theorem "for every K there is N such that whenever $G_1, G_2, \ldots, G_N$ are finite simple unlabelled graphs with $|G_i| < K + i$, there are $i < j \leq N$ such that G_i is isomorphic to a minor of G_j" still retains a big amount of unprovability/consistency strength of the infinite graph minor theorem [6].

For a survey of modern unprovability theory, see [3] or [7]. For explanations about Weiermann's phase transition programme, see [16]. For the original discussion of the metamathematics of the graph minor theorem, see [6].

We shall study the first-order graph minor theorem with growth condition from the point of view of Weiermann's phase transition programme and will try to compute exact unprovability thresholds for the graph minor theorem restricted to

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various classes of graphs. We start with an attempt to treat the case of all graphs and then move on to more manageable classes such as the class of all planar graphs.

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1. Discussion of the general case

Throughout the paper, the word 'graph' will mean a simple (without loops or parallel edges) unlabelled finite graph. The symbol $\log i$ will denote $[\log_2 i]$, the integer part of the binary logarithm of i.

For any function f, let GM_f be the statement: "for every K there is N such that for any sequence of graphs $G_1, G_2, \dots, G_N$ such that $|G_i| < K + f(i)$, there are $i < j \le N$ such that G_i is isomorphic to a minor of G_j". Using Pólya's theorem on the asymptotic of the number of graphs [8], we can provisionally conjecture that:

1. for any $r \le \sqrt{2}$, the statement $\mathrm{GM}_{r\cdot\sqrt{\log}}$ is provable in $I\Delta_0 + \exp$;
2. for any $r > \sqrt{2}$, $\mathrm{GM}_{r\cdot\sqrt{\log}}$ is unprovable in ATR_0.

The provisional conjecture comes from observing the usual behaviour of threshold functions in Weiermann's phase transition theory: the threshold function is roughly the inverse of the count-function of the investigated combinatorial class. However, graphs are different from all other combinatorial classes studied in phase transition theory so far because their count-function is faster than exponential. So, I am not excluding the possibility that the unprovability-threshold behaviour in the case of all graphs will be more complicated than what we have seen before.

Let us indeed prove $(I\Delta_0+\exp)$-provability of $\mathrm{GM}_{\sqrt{2}\cdot\sqrt{\log}}$ by an asymptotic pigeonhole argument. Let $g(n)$ be the number of non-isomorphic graphs on n vertices, $G(n)$ be $\sum_{k\le n} g(k)$. By Pólya's theorem [8],

$$g(n) \sim \frac{2^{\frac{n(n-1)}{2}}}{n!}.$$

Let us use the Stolz Lemma (see calculus textbooks, I used this one: [10], page 30) to show that

$$G(n) \sim \frac{2^{\frac{n(n-1)}{2}}}{n!}.$$

Indeed,

$$\frac{\frac{2^{n(n-1)/2}}{n!} - \frac{2^{(n-1)(n-2)/2}}{(n-1)!}}{G(n) - G(n-1)} = 1 - n\cdot 2^{1-n} \underset{n\to\infty}{\longrightarrow} 1.$$

Now, we give a usual asymptotic pigeonhole argument. Choose a number D such that for all $n \ge D$,

$$G(n) < \frac{11}{10}\cdot\frac{2^{n(n-1)/2}}{n!}.$$

Given K, choose $N > 2^{D^2}$, so that

$$\frac{11\cdot 2^{K^2/2}}{10}\cdot\frac{2^{K\sqrt{2\log N}}}{(K+\sqrt{2\log N})!\cdot 2^{(K+\sqrt{2\log N})/2}} < 1,$$

for example set $N = 2^{2^{K^2+12}} + 2^{D^2}$.

Take a sequence $G_1, G_2, \dots, G_N$ of graphs such that

$$|G_i| < K + \sqrt{2}\sqrt{\log i} \le K + \sqrt{2\log N}.$$

Notice that with this condition on growth rate, there are not enough different graphs to fill in the N spare slots in the sequence. Indeed, the number of non-isomorphic graphs of size not exceeding $K + \sqrt{2\log N}$ is

$$G(K+\sqrt{2\log N}) < \frac{11}{10}\cdot\frac{2^{(K+\sqrt{2\log N})^2/2}}{(K+\sqrt{2\log N})!\cdot 2^{(K+\sqrt{2\log N})/2}}$$

$$= \frac{11\cdot 2^{K^2/2}}{10}\cdot\frac{2^{K\sqrt{2\log N}}}{(K+\sqrt{2\log N})!\cdot 2^{(K+\sqrt{2\log N})/2}}\cdot N < N.$$

This completes the $(I\Delta_0+\exp)$-provability proof for $\mathrm{GM}_{\sqrt{2}\sqrt{\log}}$. Notice that we found not just an earlier graph isomorphic to a minor of a later graph but two copies of the same graph in this sequence, so there is no deep combinatorial reason behind the provability clause, only an asymptotic pigeonhole reason.

Remark 1. It may be possible to improve the pigeonhole argument above by counting the number of possible sequences $G_2, G_3, \ldots, G_N$ omitting the minor G_1 for all possible G_1 of size at most K.

Remark 2. We shall not concern ourselves here with the best lower bounds on the level of unprovability in our results. Throughout the paper, we prove ATR_0-unprovability but it is clear from the proof of Theorem 3 that this lower bound can be improved to any stronger theory that doesn't prove the planar graph minor theorem with growth rate condition. The author is confident that Friedman's Extended Kruskal Theorem can be deduced from it, by modifying "immersions" from [6], and hence the lower bound ATR_0 can be improved to Π^1_1-CA_0. For our purposes, we shall only need the following lemma.

Lemma 1. *For some constant B, the statement "for all k there is N such that whenever $G_1, G_2, \ldots, G_N$ are planar graphs such that for all $i \le N$, $|G_i| < k + B \cdot \log i$ then for some $i < j \le N$, G_i is isomorphic to a minor of G_j" is unprovable in* ATR_0.

In particular $\mathrm{GM}_{B\cdot\log}$ *is* ATR_0*-unprovable.*

The proof is an adaptation of immersions from [6] to this much simpler case of unordered trees without labels.

Proof. For every rooted unordered tree T, define a graph G_T as follows. The vertices of G_T are the vertices of T together with additional points d_1, d_2, d_3 and the following new vertices: for every $x \in T$, for each of its immediate successors $x_1, x_2, \ldots, x_n$ introduce new vertices $c_1, c_2, \ldots, c_n$. Clearly $|G_T| = 2 \cdot |T| + 2$. Edges are defined as follows. The root and d_1, d_2, d_3 are all connected to each other, thus forming a K_4-subgraph. For every $x \in T$, the points $x, c_1, c_2, \ldots, c_n$ are forming a circle, i.e. the edges are $xc_1, c_1c_2, \ldots, c_nx$. For every immediate successor x_i of x, x_i is connected to its corresponding new vertex c_i by an edge. In G_T, each vertex apart from the root is connected to no more than three other vertices.

Notice that for any two trees T_1 and T_2, if G_{T_1} is isomorphic to a minor in G_{T_2} then T_1 is inf-preservingly and root-preservingly embeddable into T_2.

Now, for every bad sequence of trees there is a bad sequence of planar graphs of this length, so by the Loebl–Matoušek Theorem [9], the statement "for all k there is N such that whenever $G_1, G_2, \ldots, G_N$ are planar graphs such that for all $i \le N$, $|G_i| < k + B \cdot \log i$ then for some $i < j \le N$, G_i is isomorphic to a minor of G_j" is unprovable in ATR_0 and the statement $\mathrm{GM}_{B\cdot\log}$ that implies it is also ATR_0-unprovable. $\square$

Instead of the Loebl–Matoušek theorem, we could use Weiermann's theorem from [15] and use an exact constant for Kruskal's theorem in the proof. But since we are only claiming existence of B, the Loebl–Matoušek theorem suffices.

In the case of all graphs, it may be possible to prove ATR_0-unprovability of $\mathrm{GM}_{\varepsilon\cdot\log}$ for any $\varepsilon > 0$ using the compression argument from Theorem 3 below. But since this upper bound is so far from the conjectured exact threshold, we shall not study this question here.

Remark 3. We originally hoped that it would be possible to find an appropriate approximation lemma (like Lemma 2 below) for the class of all graphs, namely to have a sequence $a_k \to_{k\to\infty} 2$ and a sequence of graphs $\langle G_k \rangle_{k\in\omega}$ such that the number of unlabelled n-vertex graphs omitting the minors $\langle G_m \rangle_{m<k}$ would be bounded below by $a_k^{n(n-1)/2}$. However this is impossible due to a phenomenon recently discovered by Norine et al. [13]: the class of all labelled graphs omitting any given minor is *small*. More precisely, for any proper minor-closed class C of labelled graphs, there is c such that for every n there are no more than $n!c^n$ labelled n-vertex graphs in C. This is of course also the upper bound on the number of unlabelled n-vertex graphs (i.e., isomorphism-types of labelled graphs) in C. Since $n!c^n$ is $o(a^{n(n-1)/2})$ for any $a > 1$, there is no hope to have this kind of approximation lemma. It has been mentioned in [2], with a sketch of the proof, that for unlabelled graphs omitting given minors, the actual bound is even smaller than $n!c^n$, namely c^n for some constant c.

So, the current state of affairs in the case of all graphs is an upper and a lower bound: $\mathrm{GM}_{\sqrt{2}\sqrt{\log}}$ is provable by the most elementary means of $I\Delta_0 + \exp$ but $\mathrm{GM}_{a\log}$ is unprovable in ATR_0 for some (possibly for all) $a > 0$. It is disappointing that the upper and the lower bounds are very far apart. However, if we restrict our class of graphs, we may be able to reach more satisfactory unprovability thresholds.

Let us fix a class of simple unlabelled graphs G. The classes we have in mind are planar graphs, connected planar graphs and graphs embeddable into a given surface. Denote the number of n-vertex members of G as g_n. The class G is said to have an unlabelled growth constant γ_G if $(g_n)^{1/n} \to_{n\to\infty} \gamma_G$.

If G is the class of all planar graphs or the class of all connected planar graphs or the class of all graphs embeddable into a given surface then γ_G exists [4] and is a number between 27.2269 and 30.061 (the same number for each of these three classes) [11]. This number is called the *unlabelled planar growth constant* and we shall denote it by γ throughout the rest of the paper.

It can be conjectured (although the author couldn't find this conjecture in the graph-theoretic sources) that every class of unlabelled graphs omitting a given set of minors has an unlabelled growth constant, as was proved in the labelled case in [1]. When this conjecture is proved, the results of this paper will generalise to all proper minor-closed classes of graphs (see discussion in the end of this article).

2. Approximation lemma

Let us first prove an approximation lemma needed for the threshold result below.

Let g_n be the number of n-vertex unlabelled planar graphs, γ be the unlabelled planar growth constant and for every $k \geq 3$, C_k be the circle on k vertices. Denote the number of n-vertex planar graphs omitting the minor C_k by $g_{n,k}$. For every n, let $f_{n,k}$ be the number of n-vertex *connected* unlabelled planar graphs omitting the minor C_k.

Throughout the rest of the paper we are going to use the following unproved conjecture $(*)$: "there is a positive number p such that for all $k \geq 3$ and all $n \geq 1$, $f_{n,k} \geq \frac{g_{n,k}}{p}$". We don't know whether this conjecture is true. Very similar statements are known to hold in the labelled case ([12], section 2), and there is a well-known conjecture by Welsh for the unlabelled case "among planar graphs, connected graphs occur with positive probability" (personal communication, but see also some discussion in [4]). We do not know how to eliminate our conjecture from Lemma 2 and, hence, from the theorems that use Lemma 2.

Lemma 2. *Assume $(*)$. Then for every $k \geq 3$, there is γ_k such that*

1. *for every $k \geq 3$, for all large enough n,* $g_{n,k} \geq (\gamma_k)^n$;
2. $\gamma = \sup_{k\in\omega} \gamma_k$.

Proof. We are going to use the superadditivity lemma (which I learnt from [12]): if $h: \mathbb{N} \to \mathbb{N}$ is such that $h(n+m) \geq h(n) \cdot h(m)$ for all $n, m \in \mathbb{N}$ then $h(n)^{1/n} \to_{n\to\infty} \sup_n (h(n))^{1/n}$.

Assuming $(^*)$, let us fix a constant $p \geq 1$ such that for every $k \geq 3$ and all $n \geq 1$,

$$f_{n,k} \geq \frac{g_{n,k}}{p}.$$

It suffices to prove superadditivity for an auxiliary function $h(n) = \frac{g_{n,k}}{p^2}$. First notice that $g_{n+m,k} \geq f_{n,k} \cdot f_{m,k}$. Indeed, for $n \neq m$, take the graph consisting of two connected components of sizes n and m. Each choice of components gives a new $(n+m)$-vertex graph. For $n = m$, consider all possible graphs consisting of two connected components of size n and all connected graphs consisting of two subgraphs of size n joined by an edge. Adding the edge doesn't spoil the property that the resulting graph omits the minor C_k because C_k is 2-connected. We counted each graph at most twice so $g_{2n,k} \geq (f_{n,k})^2$. Now,

$$h(n+m) = \frac{g_{n+m,k}}{p^2} \geq \frac{f_{n,k}}{p} \cdot \frac{f_{m,k}}{p} \geq \frac{g_{n,k}}{p^2} \cdot \frac{g_{m,k}}{p^2} = h(n) \cdot h(m).$$

Hence h is superadditive, so

$$h(n)^{1/n} \to \sup_{n\in\omega} h(n)^{1/n}.$$

Since $(p^2)^{1/n} \to_{n\to\infty} 1$, the sequence $g_{n,k}$ has the same limit. Set $\gamma_k' = \sup_n h(n)^{1/n}$.

Now, if $n < k$ then every n-vertex graph omits C_k, hence for $n < k$ we have

$$g_n = g_{n,k}.$$

So, given $\varepsilon > 0$, find N such that for all $n > N$,

$$\left(\frac{g_n}{p^2}\right)^{1/n} > \gamma - \varepsilon.$$

Notice that we have just proved that for every $k \geq 3$,

$$\left(\frac{g_{n,k}}{p^2}\right)^{1/n} \to_{n\to\infty} \sup_{n\in\omega} \left(\frac{g_{n,k}}{p^2}\right)^{1/n}.$$

Let $k > N + 1$ and notice that for all n such that $N < n < k$,

$$\left(\frac{g_{n,k}}{p^2}\right)^{1/n} = \left(\frac{g_n}{p^2}\right)^{1/n}.$$

Hence

$$\gamma - \varepsilon < \left(\frac{g_n}{p^2}\right)^{1/n} = \left(\frac{g_{n,k}}{p^2}\right)^{1/n} \leq \sup_{n\in\omega} \left(\frac{g_{n,k}}{p^2}\right)^{1/n} = \gamma_k'.$$

Hence $\gamma_k' \to_{k\to\infty} \gamma$.

Define $\gamma_k = \gamma_k' - \frac{1}{k}$. Since $\lim_{n\to\infty}(g_{n,k})^{1/n} = \gamma_k'$, for $\varepsilon = \frac{1}{k}$ there is N such that for all $n > N$,

$$(g_{n,k})^{1/n} > \gamma_k' - \varepsilon = \gamma_k.$$

So, for all $n > N$, $g_{n,k} > (\gamma_k)^n$. $\square$

This lemma can be converted into a theorem that proves a version of Theorem 16 of [1] for all unlabelled small addable classes of graphs, not just for planar graphs.

3. Threshold theorem for planar graphs

Theorem 3. *Let γ be the unlabelled planar growth constant and assume (*). Let $P(c)$ be the following statement with parameter c: "for every K there is N such that whenever $G_1, G_2, \ldots, G_N$ are unlabelled planar graphs with $|G_i| < K + c \cdot \log i$ then for some $i < j$, G_i is isomorphic to a minor of G_j". Then*

1. *for every $c \leq \frac{1}{\log \gamma}$, $P(c)$ is provable in $I\Delta_0 + \exp$;*
2. *for every $c > \frac{1}{\log \gamma}$, $P(c)$ is unprovable in ATR_0.*

It is important to understand how a 'real' parameter c can be mentioned in a first-order arithmetical formula: for each $(I\Delta_0 + \exp)$-provably recursive real number c, the statement $P(c)$ can be written using the formula that defines or approximates c.

Proof. It is easy to show the provability clause by the same asymptotic pigeonhole argument as above. As in the proof above, there will be not only an earlier graph isomorphic to a later graph but two copies of the same graph in the sequence with small growth rate.

Let us now turn to the unprovability clause. By Lemma 1, the statement "for all k there is N such that whenever $\langle S_i \rangle_{i \leq N}$ is a sequence of planar graphs with $|S_i| < k + B \cdot \log i$ then for some $i < j \leq N$, S_i is isomorphic to a minor of S_j" is unprovable in ATR_0, for some constant B.

Fix $c > \frac{1}{\log \gamma}$. Since, by Lemma 2, we have $\gamma_m \to_{m \to \infty} \gamma$, let us fix a number m such that

$$c > \frac{1}{\log \gamma_m}$$

and choose a rational number q such that

$$c > q > \frac{1}{\log \gamma_m}.$$

Put $M(n, m)$ to be $\{G \mid G \text{ is planar}, |G| \leq n \text{ and } G \text{ omits the minor } C_m\}$, so $|M(n, m)| = \sum_{\ell \leq n} g_{\ell,m}$. Find a natural number E such that for all $n \geq E$,

$$|M(n, m)| \geq (\gamma_m)^n.$$

It exists because $g_{n,m} \geq (\gamma_m)^n$ for all n from some point onwards.

Let D be large enough, so that for all $i \geq D$, we have

$$q \cdot \log i \geq E;$$
$$\gamma_m^{q \cdot \log i} \geq i;$$
$$mB \log \log i + q \log i \leq c \log i.$$

Given K, put $k = \left[\frac{K}{m+3}\right]$ and assume without loss of generality that $k \geq D$ and hence $k(m + 1) + D \leq K$.

Start off with a long bad sequence of planar graphs $S_1, S_2, \ldots, S_N$ such that $|S_i| < k + B \log i$. We shall build a new bad sequence of graphs $G_1, G_2, \ldots, G_N$ such that $|G_i| < K + c \cdot \log i$. Hence, unprovability of totality of the function defined as $k \mapsto$ longest length of a bad sequence with growth rate $k + B \log i$ implies unprovability of totality of the function $K \mapsto$ longest length of a bad sequence with growth rate $K + c \log i$.

Let $\preceq$ be any linearisation of the partial order on graphs "a graph A is isomorphic to a minor of a graph B" and let $\mathrm{enum}(n, m, i)$ be the ith element of $M(n, m)$ with respect to this ordering $\preceq$. Clearly, if $i < j$ then $\mathrm{enum}(n, m, j)$ is not isomorphic to a minor of $\mathrm{enum}(n, m, i)$.

For any graph G, let $G(C_m)$ be a new graph, obtained from G by attaching a copy of C_m to every vertex of G. (More precisely: $G(C_m)$ is the graph with domain $G \times \{0, 1, \ldots, m - 1\}$, with graph structure defined as follows: for every $a \in G$ and all $i \in \{0, 1, \ldots, m - 1\}$, the vertex $\langle a, i \rangle$ is connected by an edge to $\langle a, i + 1 \rangle$, the vertex $\langle a, m - 1 \rangle$ is connected by an edge to $\langle a, 0 \rangle$ and for any $a, b \in G$, $\langle a, 0 \rangle$ is connected by an edge to $\langle b, 0 \rangle$ if and only if a is connected by an edge to b in G.)

Build a new bad sequence of graphs as follows. For $i < D$, put G_i to be $C_{D+m-i} \cup \{w\} \cup S_1(C_m)$, where the new vertex w is identified with some vertex of C_{D+m-i} and with an arbitrary vertex of S_1 in $S_1(C_m)$ (i.e. with any vertex of the form $\langle a, 0 \rangle$, $a \in S_1$). Clearly, $|G_i| < D + m - i + km < K$. Notice that if $i < j < D$ then G_i is not isomorphic to a minor in G_j (because G_j has fewer vertices than G_i).

If $D \leq i \leq N$, let $H_i = \mathrm{enum}(q \log i, m, 2^{|i|} - i)$, where $|i|$ is the binary length of i, that is the integer part of $\log_2 i$ plus 1. It is easy to see that the function $2^{|i|} - i$ enumerates all numbers of the same binary length in reverse order.

Define G_i as $H_i \cup \{w\} \cup S_{\log i}(C_m)$ with the vertex w identified with an arbitrary vertex of H_i and with an arbitrary vertex of $S_{\log i}$ in $S_{\log i}(C_m)$ (i.e. with any vertex of the form $\langle a, 0 \rangle$, $a \in S_{\log i}$). The growth condition on G_i is satisfied since $|G_i| = |H_i| + m \cdot |S_{\log i}| - 1 < q \log i + km + mB \cdot \log \log i \leq K + c \log i$.

Let us now show that for every $i < j \leq N$, G_i is not isomorphic to any minor of G_j. Suppose there is a minor-embedding $f: V(G_i) \to V(G_j)$. (We say that $f: V(G_i) \to V(G_j)$ is a minor-embedding if f is an injection such that there is a sequence of edge-deletions, edge-contractions and deletions of isolated vertices that starts with the graph G_j and results in a minor H of G_j such that f is an isomorphism between G_i and H.)

Let us show non-embeddability for $D \le i < j$. If $\log i = \log j$ then H_i is not isomorphic to any minor in H_j, so there is $v \in H_i$ such that $f(v) \in S_{\log j} \setminus \{w\}$. Since $S_{\log i} = S_{\log j}$, by pigeonhole principle there is $u \in S_{\log i}(C_m)$ such that $f(u) \in H_j \setminus \{w\}$. But then there is a C_m-minor inside H_j, which is impossible.

Suppose that $\log i < \log j$. As before, none of the vertices of $S_{\log i}(C_m)$ can be mapped into $H_j \setminus \{w\}$, so $S_{\log i}(C_m)$ is isomorphic to a minor of $S_{\log j}(C_m)$. Let us now show that then $S_{\log i}$ is isomorphic to a minor of $S_{\log j}$. Indeed, consider our minor-embedding f of $S_{\log i}(C_m)$ into $S_{\log j}(C_m)$ and build a minor-embedding $g\colon S_{\log i} \to S_{\log j}$. Consider the image of $S_{\log i}(C_m)$ under f and contract all C_m circles in this set that are of the form $f(\{u\}(C_m))$, where $u \in S_{\log i}$. We obtained a minor in $S_{\log j}$ isomorphic to $S_{\log i}$. But, since $\log i < \log j$, $S_{\log i}$ is not isomorphic to any minor of $S_{\log j}$, so we got a contradiction.

The same argument shows that for $i < D \le j$, G_i cannot be isomorphic to a minor in G_j. Indeed, no vertex of $S_1(C_m)$ can be mapped to a vertex of $H_j \setminus \{w\}$ since H_j omits the minor C_m. Hence f minor-embeds the whole $S_1(C_m)$ into $S_{\log j}(C_m)$. In the image of $S_1(C_m)$ under f, contracting every set of the form $f(\{u\}(C_m))$ yields a minor in $S_{\log_j}$ that is isomorphic to S_1, which is impossible.

Hence $\langle G_i \rangle_{i=1}^N$ is a bad sequence we have been seeking. $\square$

Notice that although the proof above is clearly a Weiermann-style compression argument, the method of constructing a bad sequence of graphs is new, since none of the tricks used the past study of trees and sequences could be adapted here.

Theorem 4. *Let γ be the unlabelled planar growth constant and assume (*). Let $P(c)$ be the following statement with parameter c: "for every K there is N such that whenever $G_1, G_2, \dots, G_N$ are unlabelled connected planar graphs with $|G_i| < K + c \cdot \log i$ then for some $i < j$, G_i is isomorphic to a minor of G_j". Then*

1. *for every $c \le \frac{1}{\log \gamma}$, $P(c)$ is provable in $I\Delta_0 + \exp$;*
2. *for every $c > \frac{1}{\log \gamma}$, $P(c)$ is unprovable in ATR_0.*

Proof. The case $c \le \frac{1}{\log \gamma}$ follows by a usual asymptotic pigeonhole argument from the fact from [4] that if u_n is the number of unlabelled connected planar n-vertex graphs then $(u_n)^{1/n} \to \gamma$.

The case $c > \frac{1}{\log \gamma}$ follows from the fact that the reduction of Kruskal's theorem to graph minors in [6] yields unprovability of the graph minor theorem for connected planar graphs with growth rate $K + B \log i$ and that the compression argument in Theorem 3 carries through without spoiling connectivity. $\square$

Let $\mathrm{Forb}(H_1, \dots, H_n)$ be the set of all unlabelled graphs omitting the minors $H_1, \dots, H_n$.

Question 1. For which sets of graphs $\{H_1, H_2, \dots, H_n\}$ is the graph minor theorem restricted to $\mathrm{Forb}(H_1, H_2, \dots, H_n)$ unprovable? For each such set $\{H_1, H_2, \dots, H_n\}$, find the unprovability threshold for the first-order version of the graph minor theorem restricted to $\mathrm{Forb}(H_1, \dots, H_n)$.

It is sketched in [2] that the Norine–Seymour–Thomas–Wollan phenomenon transfers in full generality to the unlabelled case (namely that for every $H_1, H_2, \dots, H_n$ there are no more than d^m members of $\mathrm{Forb}(H_1, H_2, \dots, H_n)$ with m vertices). Then, if each unlabelled class omitting given minors has a growth constant and Lemma 2 (or an unlabelled version of Theorem 16 of [1]) can be proved then our Theorem 3 above will have an ultimate generalisation: for any finite set of unlabelled graphs $H_1, H_2, \dots, H_n$ as in Question 1, there is a constant $\gamma(H_1, \dots, H_n)$ such that for the statement $P_c(H_1, \dots, H_n)$ defined as "for all K there is N such that whenever $G_1, \dots, G_N$ are unlabelled simple graphs in $\mathrm{Forb}(H_1, \dots, H_n)$ and $|G_i| < K + c \cdot \log i$ then for some $i < j \le N$, G_i is isomorphic to a minor in G_j", we have

1. for all $c \le \frac{1}{\log(\gamma(H_1,\dots,H_n))}$, $P_c(H_1, \dots, H_n)$ is provable in $I\Delta_0 + \exp$;
2. for all $c > \frac{1}{\log(\gamma(H_1,\dots,H_n))}$, $P_c(H_1, \dots, H_n)$ is unprovable in ATR_0.

This ultimate future general theorem still needs some graph-theoretic work (existence of unlabelled growth constants for each $\mathrm{Forb}(H_1, \dots, H_n)$, and a version of Lemma 2 for such class) but for graphs embeddable into a given surface, the generalisation of Theorem 3 can already be proved. Indeed, notice that every planar graph is embeddable into any other surface, so the graph minor theorem restricted to graphs embeddable into a given surface is unprovable for some growth rate $K + B \log i$ for some constant B. Now, it suffices to use McDiarmid's theorem from [11] that for every surface, the class of all unlabelled graphs embeddable into this surface has the same unlabelled growth constant as the planar graphs and use Theorem 3 above for exactness. We have just proved the following theorem.

Theorem 5. *Assume (*). For any given surface S, let $P_S(c)$ be the statement "for all K there is N such that whenever $G_1, G_2, \dots, G_N$ are unlabelled graphs embeddable into the surface S with $|G_i| < K + c \log i$, there are $i < j \le N$ such that G_i is isomorphic to a minor in G_j". Then*

1. *for every $c \le \frac{1}{\log \gamma}$, $P_S(c)$ is provable in $I\Delta_0 + \exp$;*
2. *for every $c > \frac{1}{\log \gamma}$, $P_S(c)$ is unprovable in ATR_0*

where γ is the unlabelled planar growth constant.

4. Some open questions

It would be very interesting to find other classes of graphs such that the graph minor theorem restricted to these classes possesses strength. Then the count-functions for these classes will yield phase transition results between provability and unprovability for parametrised graph minor theorems for these classes. How many n-vertex graphs of tree-width k are there?

For some trivial classes (e.g. complete graphs), the graph minor theorem is trivially provable. For subcubic graphs, the graph minor theorem is unprovable [6].

In the case of multigraphs (i.e. graphs with loops and parallel edges allowed), even a rough conjecture about the logical strength of the graph minor theorem with different growth rates cannot be formulated because the number of multigraphs of size n is an open problem in graph theory (where the size of a multigraph G is defined as $|V(G)| + |E(G)|$ or in any other way monotone in $V(G)$ and $E(G)$).

Question 2. What is the strength of the statement "every countable infinite graph is a proper minor of itself"? This "Self-Minor Conjecture" conjecture due to Seymour is very strong (since it implies the infinite graph minor theorem [5], page 349) and is not known to be false. Is it strictly stronger than the infinite graph minor theorem?

Question 3. Another extremely strong statement is "countable graphs are well-quasi-ordered by minor-inclusion" [14]. Is it strictly stronger than the infinite graph minor theorem?

Question 4. Another strong statement is this. Consider the set of all minor-closed classes of graphs, ordered by the subset-relation. Is it a well-quasi-order? Find lower bounds for the logical strength of this well-quasi-orderedness assertion.

Concerning phase transitions, here is a question suggested by Weiermann about fine-tuning the threshold result.

Question 5. Replace the threshold function $\frac{1}{\log \gamma} \cdot \log i$ by the function $f(i) = (\frac{1}{\log \gamma} + \varepsilon(i)) \cdot \log i$, where $\varepsilon(i) \to_{i\to\infty} 0$. For which functions $\varepsilon(i)$ is the planar graph minor theorem with the growth rate $f(i)$ provable and for which ones unprovable?

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